Quantitative Methods

Simple and Compound Interest

Simple and Compound Interest

What we will learn in this topic……..

- **Calculate** simple and compound interest earned over multiple periods
- **Calculate** the annual compound rate given the simple rate and the frequency of compounding
- **Calculate** simple rate of interest given the annual compound rate and the frequency of compounding
- **Calculate** the effective annual rate given a nominal annual rate with continuous compounding

Simple and Compound Interest

lending money

interest

- If we postpone consumption i.e. we save, we earn a rate of return on savings.
- If we borrow i.e. take on debt, we need to pay rate of interest on borrowings.

int only on ~~original amt~~ → int on int as well.

- Let us understand concepts of simple interest and $\boxed{\text{compound interest.}}$

opportunity cost ← ~~Amt~~ calculate int only on original amt

- Let us assume rate of interest of 8% and amount invested or borrowed is $1000.

Years	Simple Interest	Compound Interest
1	1000 * 8% * 1 = 80 Amt x int rate x No. of yrs	1000 * 8% = 80 int $ for 1st yr
2	1000 * 8% * 2 = 160 → 80 + 80	80 + 1080 * 8% = 80 + 86.4 = 166.4 int $ 2nd yr
3	1000 * 8% * 3 = 240 80+80+80	166.4 + 1166.4 * 8% = 166.4 + 93.312 = 259.71 1000+166.4
4	1000 * 8% * 4 = 320 80 +80+80 +80	259.71 + 1259.71 * 8% = 259.71 + 100.78 = 360.49

Simple and Compound Interest

0 — Interest 320 — 4

1000 — 1320

- General formula to calculate value of deposit after 'T' years, that earns a particular rate of interest as per Simple and Compound interest are:

  - Simple Interest : D_0 (1+ r * T): → FV as per Simple interest
    - Hence value of deposit of 1000, at 8% after 4 years is 1000 * (1+ 8% * 4) = 1320

      Future Value of investment

      Int Amt : F.V − Original Value

      $\boxed{320}$: 1320 − 1000

  - Compound Interest : D_0 (1+ r)^T:
    - Hence value of deposit of 1000, at 8% after 4 years is 1000 * (1+ 8%)^4 = 1360.49 → FV

      $1000(1+8\%)^4$

      Int Amt : 1360.49 − 1000 = 360.49

- In the above example, we have assumed that interest is paid once a year i.e. annually.

- What if the interest is paid semi-annually (every six months)?
  - Then two adjustments are needed,
    - i) r should be divided by 2 → 'r' is annual, divide by 2
    - Ii) T should be multiplied by 2 → each yr is 2 parts

every 6m
every 3m
every m

	Simple	Comp.
	$D_0(1+r\times T)$	$D_0(1+r)^T$
For Semi annual	$D_0\left(1+\frac{r}{2}\times 2T\right)$	$D_0\left(1+\frac{r}{2}\right)^{2T}$

Simple and Compound Interest

If an investor invests \$ 5000 for 5 years @ int rate of 7%. How much money will he receive after 5 yrs. How much int will he earn?

Solve this assuming i) Simple interest ii) Compound interest

Simple

$D_0 \times (1 + r \times T) = FV$

$5000 \times (1 + 7\% \times 5) = 6750$

Int Amt: $6750 - 5000 = \boxed{1750}$

Compound

$\boxed{D_0 (1+r)^T = FV}$

$5000 \times (1 + 7\%)^5 = 7012.76$

Int Amt: $= \boxed{2012.76}$

Power of compounding

Simple and Compound Interest

- General formula to calculate value of deposit after 'T' years, and interest is paid at greater frequency than annual is (assuming compound interest):
  - Future Value of deposit : $D_0 \, (1 + r/m)^{\boxed{T*m}}$ where m is frequency of compounding
  - m = 2, 4, 12, 365 in case of semi-annual, quarterly, monthly and daily compounding respectively.

m = 2, semi-annual m = 12, monthly
m = 4, quarterly m = 365, daily

Question:

What is the value of \$100 deposit after 3 years if interest rate of 10% is compounded quarterly?

- FV of deposit = 100 * (1 + 10%/4)^(3*4) = 134.49.

$D_0 (1 + r/m)^\wedge (T \times m)$

Simple and Compound Interest

An investor invests \$4000 for 6 yrs.
what will be the future value (what will he receive) after 6 yrs? Int rate is 10%. → Stated Rate

Assume Compounded

If frequency of compounding is increased, FV ↑, int received ↑

Annually	$4000(1+10\%)^{6} = 7086.24$
Semiannually	$4000(1+10\%/2)^{12} = 7183.42$
Quarterly	$4000(1+10\%/4)^{24} = 7234.90$
~~Daily~~ Monthly	$4000(1+10\%/12)^{72} = 7270.38$

~~Actual~~ ~~Ret~~

~~Effective~~ ~~Return~~

Simple and Compound Interest

Semi annual

Eff return

$i = \left(1 + \frac{10\%}{2}\right)^2 - 1 = 10.25\%$

Quarterly

$= \left(1 + \frac{10\%}{4}\right)^4 - 1 = 10.38\%$

- **Effective annual interest rate**:
- When interest rate is annually compounded, rate stated for the deposit and effective rate is the same. 10% annual compounding means return is 10% per year.

when compounding is done more than once annually, Eff rate of return > ~~Act~~ stated rate

- What if the interest is compounded semi-annually ?
  - Then stated rate of 10% and effective rate of interest (that investor receives) is different.
  - The formula for effective rate of interest is: $i = (1 + r/m)^m - 1$, where i is effective rate of interest, r is the stated annual rate also called annual percentage rate and m is frequency of compounding.

APR

  - Given effective rate, we can work out, annual percentage rate in reverse way by using following formula:
    - $r = m * ((1 + i)^{1/m} - 1)$.

Monthly: $\left(1 + \frac{10\%}{12}\right)^{12} - 1 = 10.47\%$

Daily: $\left(1 + 10\%/365\right)^{365} - 1 = 10.51\%$

Simple and Compound Interest

The effective rate of interest assuming semi-annual compounding ~~to~~ is 10.25%. How much is APR or stated annual rate?

$$\boxed{i = \left(1+\frac{r}{m}\right)^m - 1}$$

$i \rightarrow$ Effective int rate

$r \rightarrow$ stated annual rate

$$10.25\% = \left(1+\frac{r}{2}\right)^2 - 1$$

$$1.1025 = \left(1+r/2\right)^2$$

$$1.1025^{1/2} = 1+r/2$$

$$1.05 - 1 = r/2$$

$$\cancel{0.05}\ 0.05 = r/2$$

$$0.05 \times 2 = r$$

10%

Simple and Compound Interest

The effective rate of interest assuming quarterly compounding ~~to~~ is 11.46%. How much is APR or stated annual rate?

$$\boxed{i = \left(1+\frac{r}{m}\right)^{m} - 1}$$

i → Effective int rate

r → stated annual rate

$$11.46\% = \left(1+\frac{r}{4}\right)^4 - 1$$

$$1.1146 = \left(1+r/4\right)^4$$

$$1.1146^{1/4} = 1+r/4$$

$$1.0275 = 1+r/4$$

$$0.0275 = r/4$$

$$11\% = r$$

Simple and Compound Interest

- **Continuous compounding**
- We can keep reducing interval of compounding from annual to semi –annual to quarterly to daily and then to continuous compounding.

- What is the FV of deposit if interest is continuously compounded?
  - FV of Deposit: $D_T = D_0\, e^{rT}$, e is exponential constant, approx. = 2.718.

  - What is the FV of deposit of $1000 after 25 years assuming interest is compounded continuously at 8%?
    - FV = 1000 * $e^{0.08*25}$ = $7389.06.

$$D_T = D_0 \exp^{r.T}$$

r = rate of int conti comp.

T = No. of yrs.

Simple and Compound Interest

Question:

- A credit card company charges $\boxed{\text{2\% interest monthly}}$ on outstanding credit card balances. How much is annual percentage rate?

Annual % rate

Approx: $2\% \times 12 = 24\%$

Actual: Effective: $(1+2\%)^{12} - 1 = \boxed{26.82\%}$

00.00

every month 2%

$\boxed{\text{2\% int monthly}}$

Not same

2% annual compounded monthly

$2/12$

Time Value of Money

Time value of money

Learning Outcomes........

- **Calculate and Interpret** future values for single sums and annuities
- **Calculate and Interpret** present values for single sums, annuities and perpetuities
- **Calculate** equal instalments on a repayment mortgage given the present value of the borrowings, the fixed mortgage rate and the term of the borrowing

Future Value and Present Value : Single Sum

- Future value of single sum: $FV = PV * (1 + r)^T$

Diagram:

FV?

→ Time line

Today ← T=0

T=T

PV → Money invested

Money Back

Example:

- What is the future value of QAR 75000 invested for 12 years at the rate of 7% compounded annually?

PV r

$$75000 \times (1 + 7\%)^{12} = 168914.37$$

- What is the future value in above case if interest is compounded semi-annually?

$$75000 \times \left(1 + \frac{7\%}{2}\right)^{(12\times 2)} = 171249.64$$

Time value of money

$FV = PV * (1+r)^T$

$\frac{FV}{(1+r)^T} = PV$

Future Value and Present Value : Single Sum

➢ Present Value of single sum: $\boxed{PV = FV / (1 + r)^T}$

$PV = \frac{1}{(1+r)^T}$

Diagram:

T=0 T

PV ? Discounting FV (Given)

➢ PV Factor or Discount Factor = $1 / (1 + r)^T$

→ FV = $1

➢ The concept of PV is very critical in financial markets, since **value of any asset** such as stock or bond today is its PV.

➢ PV is calculated by discounting future cashflows.

Calculate PV factor given int rate of 5% & yrs of 8.

PV factor = $\frac{1}{(1+5\%)^8}$

Example:

➢ What is the PV of following investments? Assume appropriate interest rate is 10% for both the investments?

- ➢ Investment A: Pays $4000 after 4 years → $4000/(1+10\%)^4 = 2732.05$
- ➢ Investment B: Pays $5200 after 6 years → $5200/1.1^6 = 2935.26$

Time value of money

Future Value and Present Value : Annuity and Mortgages

investing

- In financial world, there are many investments that involve regular contributions or withdrawals till maturity.

taking money out

2 yrs

1 yr

1000 1000 1000 No interest

0 1 2

FV ?

- How does the mathematics work for this?

Example:

- Suppose an investment plan involves contributions of D_0, d_1 and d_2 now, at the end of first year and at the end of second year, respectively. If rate of interest is 'r', what will be the value of this investment at the end of two years?

- Value of investment at the end of 1st year assuming contribution at the end of 1st year is made: $D_1 = D_0*(1 + r)^1 + d_1$

- Value of investment at the end of ~~1st~~ 2nd year assuming contribution at the end of ~~1st~~ 2nd year is made: $D_2 = D_0*(1 + r)^2 + d_1*(1 + r)^1 + d_2$

Time value of money

d_0 d_1 d_2

1000 1000 1000

0 1 2

FV at end yr 1 = $1000(1+r)^1 + 1000$

FV at end yr 2 = $1000(1+r)^2 + 1000(1+r)^1 + 1000$

Future Value and Present Value : Annuity and Mortgages

- What is all contributions or withdrawals (d_1, d_2 , d_3) are of same amount? How will the formula change?

- Assuming the withdrawals / contributions are same the formula is

$+d \rightarrow$ contribution

$-d \rightarrow$ withdrawal

We can then show that the value of the investment after T years is:

$$\boxed{\underbrace{D_T}_{\text{FV}} = \underbrace{D_0\,(1+r)^T}_{\text{FV of } D_0} + \underbrace{\frac{d(1+r)^T - d}{r}}_{\text{Annuity}}}$$

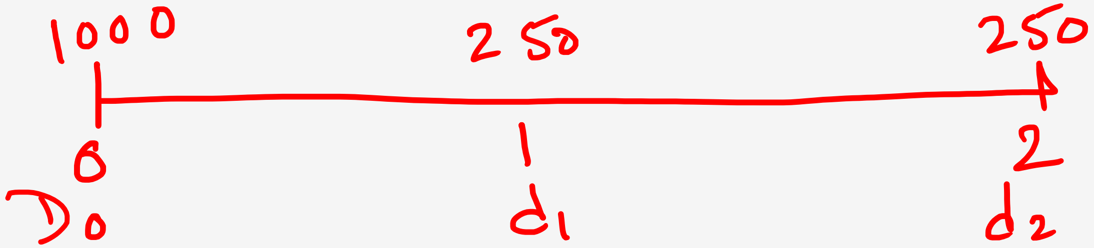
Annuity $\rightarrow$ regular investment/contribution

$\rightarrow$ regular withdrawal

Time value of money

Future Value and Present Value : Annuity and Mortgages

- Example: What is the future value of investment of $1000 made today at the rate of 10% after two years. Assume that additional contributions of $250 are made at the end of each year for two years?
- Diagram:



$d_1 = d_2 = d$

- Detailed working: FV = 1000 * (1+10%)^2 + 250 * (1+10%)^1 + 250 = 1735
- Formula: 1000* (1+10%)^2 + (250 *(1+10%)^2 – 250)/0.1 = 1735

$$\underbrace{D_0 \times (1+r)^T} + \big(\underbrace{d \times (1+r)^T - d}\big)/r =$$

Time value of money

We can then show that the value of the investment after T years is:

$$D_T = D_0\,(1+r)^T + \frac{d(1+r)^T - d}{r}$$

An investor invested \$2500 now. At the end of each year for next 3 yrs, he invested \$1500. How much is FV given int rate of 8%.

$$2500\,(1+8\%)^3 + \frac{1500(1.08)^3 - 1500}{0.08}$$

$$= 8018.88$$

$$D_0 = \frac{-d(1+r)^T + d}{r} \times \frac{1}{(1+r)^T}$$

A person would like to withdraw 5000 at the end of each yr for 10 yrs. How much he should invest today to buy this annuity? Rate of interest is 6%?

$$\frac{-5000(1.06)^{10} + 5000}{0.06} \times \frac{1}{1.06^{10}}$$

$$= -36800.44$$

Time value of money

Future Value and Present Value : Annuity and Mortgages

- **Annuity**: In case of annuity (which is usually a sum paid to an insurance company), fixed number of withdrawals take place for a number of years at the end of which the value of investment becomes zero. We can adjust the formula used earlier as follows:

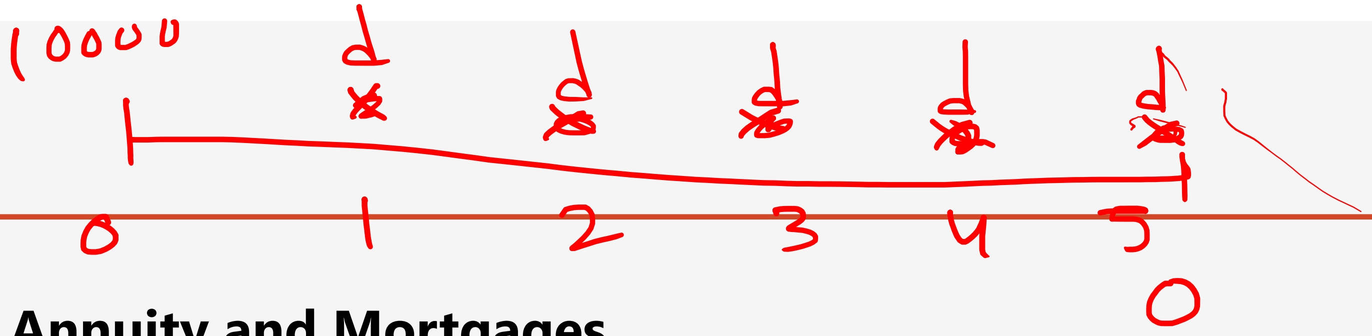
Age → 30 yrs
Work → another 25 yrs
Retire 55 yrs

$$D_0 = \frac{-d(1+r)^T + d}{r} \times \frac{1}{(1+r)^T}$$

$\boxed{\$\ 5\ \text{mln}}$ → Annuity
25 yrs

Example:
- A retired person wants to purchase an annuity that pays him $2000 at the end of each year for next 5 years. If the rate of interest is 10%, how much he should pay for this annuity now?

- $D_0 = (-2000*(1+0.1)^5 + 2000)/0.1 * 1/(1+0.1)^5 = 7{,}582.$



Future Value and Present Value : Annuity and Mortgages

- **Mortgage**: In case of mortgage, the borrower pays instalments regularly in such as way that at the end of mortgage life, the entire loan is repaid, and it becomes zero.
- We can use the following formula to calculate the instalment amount:
  - $d = D_0 * (1 + r)^T / [((1 + r)^T – 1)/ r]$
  - Suppose a loan of 50,000 is to be paid over 25 years in equal instalments and the rate of interest is 8%, how much will be the loan instalment?
  - Instalment = $50{,}000 * (1+ 0.08)^{25} / [((1+ 0.08)^{25} -1) / 0.08] = 4{,}683.94$

1) FV of multiple payments
2) Annuity Value
3) Mortgage payment

Future Value and Present Value : Annuity and Mortgages No maturity

- **Perpetuity**: This represents a constant flow of income indefinitely or perpetually,
- The PV of Perpetuity can be expressed as = C/r, where C is the constant sum and r is the discount rate.

Suppose a perpetuity pays £ 7000 per yr & investor requires rate of return of 5%, what is PV of perpetuity?

$$PV = \frac{7000}{0.05} =$$